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Crack kinking from an initially closed crack

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Abstract

Crack kinking in elastic solids in two-dimensional situations is studied in the case where the crack is initially closed (without friction) due to compressive forces but kinking opens it. The first problem which arises, namely the determination of the stress expansion near the crack tip prior to kinking has been studied previously showing that because of contact between the crack lips, the classical singular mode I term of this expansion is replaced by another, bounded one involving a new non-singular stress corresponding to a uniform compressive stress perpendicular to the crack lips. From there, one derives *universal* formulae for the first two terms of the expansion of the stress intensity factors at the tip of the open, extended crack in powers of the crack extension length. Combining the formulae found with the *principle of local symmetry* one can then determine first the kink angle, which is found to always amount to precisely $\pm 77.3^\circ$, the sign being opposite to that of the initial mode II stress intensity factor, and second the initial curvature of the crack extension, which is found to depend upon both initial non-singular stresses. The problem of whether or not, after the initial kink, the crack tends to come back to its original direction is finally investigated. © 1999 Elsevier Science Ltd. All rights reserved.

Keywords: Crack kinking; Initially closed crack; Frictionless contact; Stress intensity factors; Non-singular stresses; Principle of local symmetry

1. Introduction

A vast literature has been devoted to the prediction of crack paths in elastic solids in two-dimensional situations subjected to arbitrary loadings, and in particular to crack kinking in the presence of mode II, assuming absence of contact between the crack lips. Some relevant references are Goldstein and Salganik (1974), Bilby and Cardew (1975), Chatterjee (1975), Wu (1978a, 1978b, 1979), Amestoy et al. (1979), Cotterell and Rice (1980), Karihaloo et al. (1981), Ichikawa and Tanaka (1982), Sumi et al. (1983), Sumi (1986), Amestoy (1987), Leblond (1989), Amestoy and Leblond (1992), Leguillon (1993).

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Here we consider the same problem but allowing for unilateral *frictionless* contact between the crack lips. It is assumed that *due to some compressive forces, the crack lips are initially in contact in the vicinity of the crack tip*, but that *crack kinking occurs so as to open the crack, at least near its tip*. The problem investigated is to study the shape of the beginning of the crack extension (kink angle, initial curvature).

The first, preliminary but indispensable task is to determine the form of the stress expansion in powers of the distance to the initial crack tip in the initial situation where the crack is closed by compressive forces (in the absence of friction). This has been done by Deng (1994). This author showed that due to unilateral contact, the singular (divergent) mode I term disappears from the stress expansion, together with the corresponding stress intensity factor K_{I}. (The singular mode II term proportional to K_{II}, on the other hand, remains unchanged). The singular mode I term is replaced by some new, bounded term involving a *second non-singular stress* T_{II} representing a uniform compression stress field perpendicular to the crack lips. (The other non-singular term, involving the usual first non-singular stress $T \equiv T_{\mathrm{I}}$ representing a uniform tension or compression stress field parallel to the crack lips, is unaffected).

The next task is to derive the expansion of the stress intensity factors $K_{\mathrm{I}}(s)$, $K_{\mathrm{II}}(s)$ at the tip of the open crack extension in powers of the crack extension length s. This is done in two steps. The first one consists of deriving the expression of the first two terms of this expansion in terms of the initial stress intensity factor K_{II}, the two initial non-singular stresses T_{I}, T_{II}, the kink angle φ and the curvature parameter a of the crack extension. The arguments used are essentially based on homothetical transformations and homogeneity properties, just like in the work of Leblond (1989) in the absence of contact. The formulae obtained involve some 'universal' functions depending only on the kink angle φ. The procedure does not immediately provide the values of these functions, but does indicate how to calculate them through suitable finite element computations (including of course frictionless contact). These computations are performed, in a second step, using the CASTEM finite element code.

It only remains then to combine the expansion of the second stress intensity factor $K_{\mathrm{II}}(s)$ in powers of s with Goldstein and Salganik's (1974) well-known and widely accepted *principle of local symmetry* which stipulates that $K_{\mathrm{II}}(s)$ must be identically zero all along the actual (open) propagation path, to obtain the values of the kink angle φ and the curvature parameter a of the crack extension. It is found that because of the absence of an initial K_{I}, the kink angle is uniquely determined by the sign of K_{II}: $\varphi = -77.3^{\circ}$ for $K_{\mathrm{II}} > 0$ and $+77.3^{\circ}$ for $K_{\mathrm{II}} < 0$. On the other hand, the curvature parameter a depends on the initial stress intensity factor K_{II} and both initial non-singular stresses T_{I}, T_{II}.

It is interesting to compare the signs of φ and a since this comparison says whether or not after the initial kink, the crack tends to further deviate from its initial direction. It is found that the relative signs of φ and a depend on the initial non-singular stresses T_{I} and T_{II}. For $T_{\mathrm{II}} = 0$, a positive (negative) T_{I} results in identical (different) signs for φ and a, and thus in a crack extension which further deviates from (comes back to) the initial crack direction. These conclusions concerning the influence of T_{I} are analogous to those arrived at by Cotterell and Rice (1980) in the case without contact. For $T_{\mathrm{I}} = 0$, a negative T_{II} (positive values being ruled out because of initial contact between the crack lips) generates identical signs for φ and a, so that the crack tends to further deviate from its initial direction. This result is completely new and specific to the case where contact between the crack lips occurs initially, since there is no such quantity as T_{II} in the absence of contact.

2. Preliminaries

2.1. Presentation of the problem

Figure 1 schematically depicts the problem investigated. An isotropic elastic body Ω of arbitrary shape is loaded in plane strain conditions through some constant (in time) displacements $\mathbf{u}^p$ prescribed

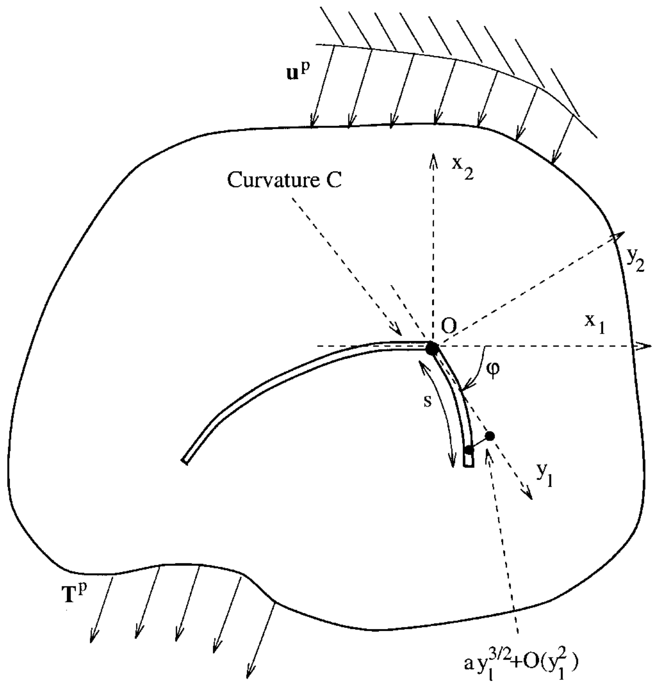

Fig. 1. The general problem studied.

on the portion $\partial\Omega_u$ of its boundary and constant tractions $\mathbf{T}^p$ prescribed on the complementary part $\partial\Omega_T$[1]. This body contains an initial crack of arbitrary shape, the tip of which is denoted O. The curvature of this initial crack at the point O is C. Let Ox_1x_2 denote the frame 'adapted' to the crack at the point O. The crack extends in a direction which makes an initial angle φ (the *kink angle*) with the original tangent Ox_1 at O. Let Oy_1y_2 denote the frame obtained from Ox_1x_2 by rotating it by the angle φ. The shape of the crack extension, the length of which is denoted s, is described by the equation

$$y_2 = ay_1^{3/2} + O(y_1^2) \tag{1}$$

where a is a 'curvature parameter'. In fact, eqn (1), with $a \neq 0$, implies that the curvature of the crack extension is infinite at the point O. The necessity of considering such singular shapes of the crack extension was established in many papers, notably Cotterell and Rice (1980), in the absence of contact, and will also be apparent below when contact is present. The problem investigated will be to *predict the values of the kink angle φ and the curvature parameter a, as functions of the loading*.

2.2. The stress expansions near the initial and final crack tips

The first question which arises is that of the asymptotic form of the stress field near the crack tip. In the absence of contact, it is of the well-known form

[1] It can be shown that introducing a time-dependence of the loading would not modify the conclusions drawn about the path followed by the crack, under the assumption of a proportionally varying loading; the proof is the same as in the absence of contact (see Amestoy and Leblond, 1992).

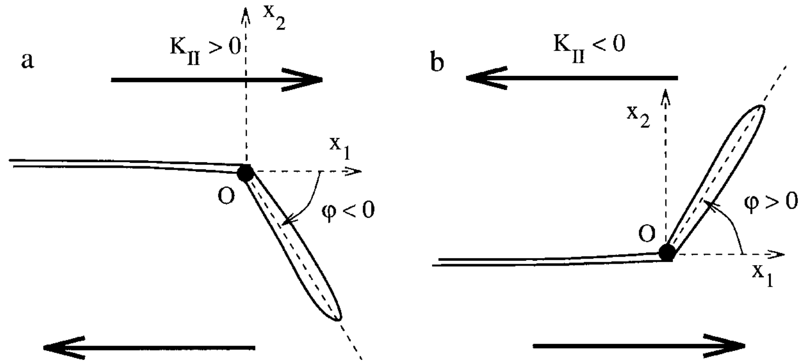

Fig. 2. The two cases where the crack extension is open: (a) $K_{\mathrm{II}} > 0$, $\varphi < 0$; (b) $K_{\mathrm{II}} < 0$, $\varphi > 0$.

$$\sigma_{ij}(r,\theta) = K_{\mathrm{I}}\frac{f_{\mathrm{I}ij}(\theta)}{\sqrt{r}} + K_{\mathrm{II}}\frac{f_{\mathrm{II}ij}(\theta)}{\sqrt{r}} + Tg_{ij}(\theta) + O\left(\sqrt{r}\right) \tag{2}$$

for 'adapted' polar coordinates r, θ, where K_{I} and K_{II} are the stress intensity factors of modes I and II and T the non-singular stress, corresponding to some uniform tension or compression stress field parallel to the tangent to the crack at its tip (which does not 'see' the crack) ($T \equiv \sigma_{\tau\tau} \equiv \boldsymbol{\tau}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{\tau}$ where $\boldsymbol{\tau}$ denotes the local unit tangent vector to the crack). Eqn (2) applies in particular *at the tip of the extended crack*, since kinking is assumed to result in an *open* crack extension, provided of course that K_{I}, K_{II} and T are interpreted as those quantities $K_{\mathrm{I}}(s)$, $K_{\mathrm{II}}(s)$, $T(s)$ after propagation over a distance s. But it *does not apply to the initial situation* where the crack is closed in the vicinity of its initial tip O. It has been shown by Deng (1994) (in the presence of friction, but here we apply his result without it) that the stress expansion then takes the form

$$\sigma_{ij}(r,\theta) = K_{\mathrm{II}}\frac{f_{\mathrm{II}ij}(\theta)}{\sqrt{r}} + T_{\mathrm{I}}g_{\mathrm{I}ij}(\theta) + T_{\mathrm{II}}g_{\mathrm{II}ij}(\theta) + O\left(\sqrt{r}\right) \tag{3}$$

where $K_{\mathrm{II}}f_{\mathrm{II}ij}(\theta)/\sqrt{r}$ still represents the singular mode II field, but where the singular mode I term has disappeared. On the other hand, the non-singular, bounded term now comprises contributions arising from *two* (instead of just one) non-singular stresses, $T_{\mathrm{I}} \equiv T$ which still represents a uniform tension or compression stress field parallel to the tangent to the crack at its tip, and T_{II} which represents a *uniform compression stress field perpendicular to this tangent* ($T_{\mathrm{II}} \equiv \sigma_{nn} \equiv \mathbf{n}\cdot\boldsymbol{\sigma}\cdot\mathbf{n}$ where $\mathbf{n}$ denotes the local unit normal vector to the crack); T_{II} is necessarily *negative* since contact is assumed to take place. Although Deng's (1994) rigorous proof is of course quite welcome, eqn (3) could easily be foreseen by noting that in the presence of contact, the (compressive) mode I field does not 'see' the crack any longer and thus *must* become non-singular.

It is clear in Fig. 2 that if the initial stress intensity factor K_{II} is positive, φ must be negative for the crack extension to be open; conversely, if $K_{\mathrm{II}}<0$, one must have $\varphi > 0$. Furthermore, the two situations are symmetrical. *We shall therefore just consider the case where* $K_{\mathrm{II}} > 0$ *and* $\varphi<0$ *in the sequel*, except at specific places where we shall indicate what the results become for $K_{\mathrm{II}}<0$ and $\varphi > 0$.

2.3. Homogeneity property of the type of problem considered

It is important to note that the problems of elastic fracture mechanics with unilateral frictionless contact between the crack lips considered in this paper, though of course non-linear, are *positively* *homogeneous of degree* 1; in other words, one can obtain a new solution from an old one by multiplying all displacements and stresses by some positive factor λ. Indeed, the conditions of frictionless unilateral contact between the crack lips read, at the generic point of the crack:

$$\sigma^+_{\tau n} = \sigma^-_{\tau n} = 0 \quad \text{and} \quad \begin{cases} \sigma^+_{nn} = \sigma^-_{nn} = 0 \quad \text{and} \quad [\![u_n]\!] > 0 \\ \text{or} \\ \sigma^+_{nn} = \sigma^-_{nn} < 0 \quad \text{and} \quad [\![u_n]\!] = 0 \end{cases}$$

where $\sigma_{\tau n} \equiv \boldsymbol{\tau} \cdot \boldsymbol{\sigma} \cdot \mathbf{n}$, $\boldsymbol{\tau}$ and $\mathbf{n}$ denoting the unit local tangent and normal vectors to the crack as above, and $[\![u_n]\!] \equiv \mathbf{u}^+ \cdot \mathbf{n} - \mathbf{u}^- \cdot \mathbf{n}$ is the discontinuity of normal displacement across the crack. It is clear that if these conditions are satisfied on the whole crack for some fields $\mathbf{u}$ and $\boldsymbol{\sigma}$, they are also satisfied by the fields $\lambda\mathbf{u}$ and $\lambda\boldsymbol{\sigma}$ for $\lambda > 0$. The result follows from there and the linearity of the equations of elasticity.

One might be surprised by this property since it does not hold for such classical contact problems as that of Hertz for instance. The origin of the difference is as follows. In the Hertz problem, the points which come into contact are separated initially, so that the contact conditions involve the initial distance between these points. Thus they are violated if one changes the displacement field without modifying the (initial) geometry. Here, the points of the crack lips which come into contact are already confounded initially. Thus there appears no such thing as the initial distance between the points in the contact conditions, which is why they remain satisfied if both the displacement and stress fields are multiplied by a positive constant.

3. The stress intensity factors just after the kink

The object of investigation of this section is the first term of the expansion of the stress intensity factors at the tip of the extended crack in powers of the crack extension length s, or in other words, the stress intensity factors K^*_{I}, K^*_{II} just after the kink:

$$K^*_{\mathrm{I}} \equiv \lim_{s\to 0} K_{\mathrm{I}}(s); \quad K^*_{\mathrm{II}} \equiv \lim_{s\to 0} K_{\mathrm{II}}(s).$$

More specifically, we shall prove (with a reasonable assumption that will be stated below) that whatever the geometry and the loading, K^*_{I} and K^*_{II} *depend solely on the stress intensity factor* K_{II} *of the initially closed crack and the kink angle* φ (linearly with respect to K_{II} with the hypothesis $K_{\mathrm{II}} > 0$ stated above). This result is the equivalent, in the case where the crack lips are initially in contact, of that derived by Leblond (1989) in the absence of contact and later confirmed by Leguillon (1993), namely that in that case K^*_{I} and K^*_{II} depend only on the initial stress intensity factors K_{I}, K_{II} plus the kink angle φ (linearly with respect to K_{I} and K_{II}).

We shall use the vectorial notations

$$\mathbf{K}(s) \equiv (K_{\mathrm{I}}(s), K_{\mathrm{II}}(s)); \quad \mathbf{K}^* \equiv \left(K^*_{\mathrm{I}}, K^*_{\mathrm{II}}\right); \tag{4}$$

thus

$$\mathbf{K}^* \equiv \lim_{s\to 0} \mathbf{K}(s). \tag{5}$$

Let us first suppose that the body considered is a circular disk of centre O (the initial crack tip where kinking occurs), of radius R, containing an *edge*[2] crack and subjected to some prescribed traction field

[2] So that there is no need to specify the length of the principal branch of the crack for a full geometric description of the problem.

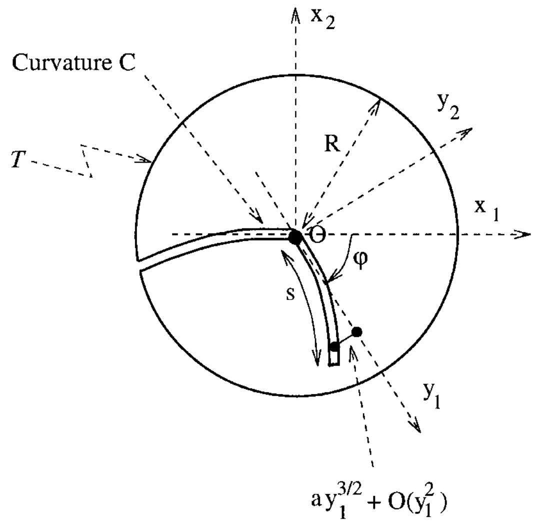

Fig. 3. The particular case of a circular disk.

$\mathscr{T}$ on its boundary (Fig. 3). The vector of stress intensity factors $K_{\mathrm{I}}(s)$, $K_{\mathrm{II}}(s)$ at the tip of the extended crack can be symbolically expressed as

$$\mathbf{K}(s) \equiv \mathbb{L}(\varphi, R, C, a, s; \mathscr{T}) \tag{6}$$

where $\mathbb{L}$ is a (vectorial) functional, depending on the geometric parameters φ, R, C, a, s, of the traction field $\mathscr{T}$, positively homogeneous of degree 1 (but of course non-linear) with respect to that field:

$$\mathbb{L}(\varphi, R, C, a, s; \lambda\mathscr{T}) = \lambda\mathbb{L}(\varphi, R, C, a, s; \mathscr{T})\ (\forall\lambda > 0) \tag{7}$$

since it has been noted above that problems of the type considered are positively homogeneous of degree 1.

Let us now multiply *both the distances* (change of scale) *and the displacements* by some positive factor λ, while keeping the strains and stresses (and therefore the surface tractions) unchanged. We thus obtain a new solution to the equations of elastic fracture mechanics with contact. In this operation, the stress intensity factors at the tip of the extended crack, being limits of certain stress components times the square root of a vanishingly small distance, are multiplied by $\sqrt{\lambda}$. Since in the homothetical transformation considered, the parameters φ, R, C, a, s become $\varphi, \lambda R, C/\lambda, a/\sqrt{\lambda}, \lambda s$ respectively, it follows that the functional $\mathbb{L}$ verifies the following 'positive homogeneity' property with respect to its geometric arguments:

$$\mathbb{L}\left(\varphi, \lambda R, C/\lambda, a/\sqrt{\lambda}, \lambda s; \mathscr{T}\right) = \sqrt{\lambda}\mathbb{L}(\varphi, R, C, a, s; \mathscr{T})\ (\forall\lambda > 0). \tag{8}$$

Now let $\mathbb{L}^*(\varphi, R, C, a; \mathscr{T})$ denote the limit of the functional $\mathbb{L}(\varphi, R, C, a, s; \mathscr{T})$ for $s \to 0$ (this is the functional that gives the stress intensity factors K_{I}^*, K_{II}^* just after the kink). Just like $\mathbb{L}$, $\mathbb{L}^*$ is positively homogeneous of degree 1 with respect to the loading $\mathscr{T}$:

$$\mathbb{L}^*(\varphi, R, C, a; \lambda\mathscr{T}) = \lambda\mathbb{L}^*(\varphi, R, C, a; \mathscr{T})\ (\forall\lambda > 0) \tag{9}$$

as is evident by taking the limit $s \to 0$ in eqn (7). Taking the same limit in eqn (8), one also obtains a 'positive homogeneity' property for $\mathbb{L}^*$ with respect to its geometric arguments analogous to that for $\mathbb{L}$:

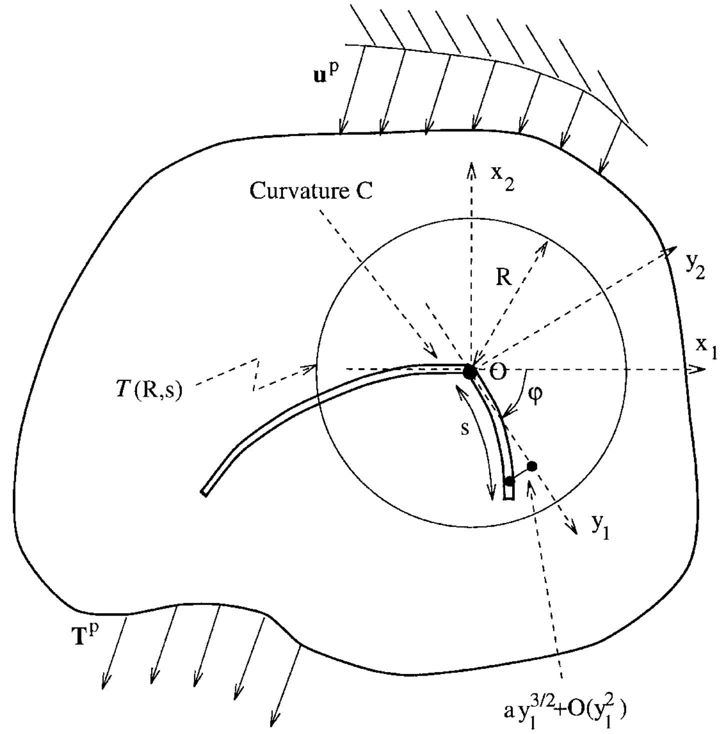

Fig. 4. Circular disk centered at the original crack tip in an arbitrary body.

$$\mathbb{L}^*\left(\varphi, \lambda R, C/\lambda, a/\sqrt{\lambda}; \mathscr{T}\right) = \sqrt{\lambda}\mathbb{L}^*(\varphi, R, C, a; \mathscr{T}) \ (\forall \lambda > 0). \tag{10}$$

We now come back to the general case of a body of arbitrary shape (Fig. 4). We consider, within this body, circular disks of centre O and sufficiently small radius R for the crack to intersect their boundary, i.e. to be an edge crack within them. Let $\mathscr{T}\ (R, s)$ denote the traction field exerted on the boundary of the disk of radius R, when the length of the crack extension is s, as a result of the application of the prescribed displacements $\mathbf{u}^p$ on $\partial\Omega_u$ and prescribed tractions $\mathbf{T}^p$ on $\partial\Omega_T$. The mechanical fields inside the disk of radius R, and therefore the stress intensity factors at the tip of the extended crack, are unchanged if one eliminates the exterior of this disk while preserving the traction field $\mathscr{T}\ (R, s)$ exerted on its boundary. Thus formula (6) applies and yields

$$\mathbf{K}(s) \equiv \mathbb{L}\big(\varphi, R, C, a, s; \mathscr{T}(R, s)\big). \tag{11}$$

We now introduce the assumption that *the stresses at a given, fixed point are continuous and differentiable with respect to s at the point* $s = 0$. These properties were established in a fully rigorous way in the absence of contact by Leblond (1989). The proofs unfortunately cannot be easily extended to account for the possibility of contact. This assumption is quite reasonable, however. It implies that the traction field $\mathscr{T}\ (R, s)$, for a fixed R, is of the form, for $s \to 0$:

$$\mathscr{T}(R, s) \equiv \mathscr{T}(R) + O(s) \tag{12}$$

where $\mathscr{T}\ (R)$ denotes the traction field exerted on the boundary of the disk of radius R *prior to kinking and propagation of the crack.*

With that assumption, eqn (11) becomes, in the limit $s \to 0$:

$$\mathbf{K}^* \equiv \lim_{s \to 0} \mathbf{K}(s) = \mathbb{L}^*\big(\varphi, R, C, a; \mathscr{T}(R)\big), \tag{13}$$

which shows that the stress intensity factors *just after* the kink depend on the mechanical fields only through their values *prior to kinking*. It only remains to take the limit $R \to 0$ in eqn (13) to get the desired result. Prior to doing that, however, we transform this formula, using eqn (10) with $\lambda = 1/R$, then eqn (9) with $\lambda = \sqrt{R}$, into

$$\mathbf{K}^* = \sqrt{R}\mathbb{L}^*\big(\varphi, 1, RC, \sqrt{R}a; \mathscr{T}(R)\big) = \mathbb{L}^*\big(\varphi, 1, RC, \sqrt{R}a; \sqrt{R}\mathscr{T}(R)\big). \tag{14}$$

Now, by eqn (3), the surface traction corresponding to the field $\mathscr{T}(R)$ is of the form

$$\mathrm{T}(R, \theta) \equiv \frac{K_{\mathrm{II}}}{\sqrt{R}}\mathbf{f}_{\mathrm{II}}(\theta) \cdot \mathbf{e}_r(\theta) + O(1)$$

where $\mathbf{e}_r$ denotes the unit radial vector, so that

$$\sqrt{R}\mathscr{T}(R) \equiv K_{\mathrm{II}}\big\{\mathbf{f}_{\mathrm{II}}(\theta) \cdot \mathbf{e}_r(\theta)\big\} + O\big(\sqrt{R}\big) \tag{15}$$

where $\{\mathbf{T}(\theta)\}$ denotes the field defined by the surface traction $\mathbf{T}(\theta)$. Inserting eqn (15) into eqn (14_2), taking the limit $R \to 0$ (which is licit since eqn (14) holds for all sufficiently small values of R) and using eqn (9) with $\lambda = K_{\mathrm{II}}$ (which is recalled to be assumed to be positive, as required by this equation), we get

$$\mathbf{K}^* = \mathbb{L}^*\big(\varphi, 1, 0, 0; K_{\mathrm{II}}\big\{\mathbf{f}_{\mathrm{II}}(\theta) \cdot \mathbf{e}_r(\theta)\big\}\big) = K_{\mathrm{II}}\mathbb{L}^*\big(\varphi, 1, 0, 0; \big\{\mathbf{f}_{\mathrm{II}}(\theta) \cdot \mathbf{e}_r(\theta)\big\}\big) \tag{16}$$

or, in component form,

$$\begin{cases} K_{\mathrm{I}}^* \equiv \tilde{F}_{\mathrm{I,II}}(\varphi)K_{\mathrm{II}} \\ K_{\mathrm{II}}^* \equiv \tilde{F}_{\mathrm{II,II}}(\varphi)K_{\mathrm{II}} \end{cases} \tag{17}$$

where $\tilde{F}_{\mathrm{I,II}}(\varphi)$ and $\tilde{F}_{\mathrm{II,II}}(\varphi)$ denote functions depending only on φ and not on curvature parameters (the commas here do not denote a differentiation, they only serve to clearly separate the indices). This result is to be compared to that obtained by Leblond (1989) in the absence of contact, namely

$$\begin{cases} K_{\mathrm{I}}^* \equiv F_{\mathrm{I,I}}(\varphi)K_{\mathrm{I}} + F_{\mathrm{I,II}}(\varphi)K_{\mathrm{II}} \\ K_{\mathrm{II}}^* \equiv F_{\mathrm{II,I}}(\varphi)K_{\mathrm{I}} + F_{\mathrm{II,II}}(\varphi)K_{\mathrm{II}} \end{cases} \tag{18}$$

where the $F_{p,q}(\varphi)$ are analogous functions depending only on φ. In both cases (with or without contact initially), all these functions are *universal* in the sense that they apply to any situation, whatever the geometry and the loading.

4. Second term of the expansion of the stress intensity factors in powers of the crack extension length

We now wish to study the form of the second term, which will be seen to be proportional to $\sqrt{s}$, of the expansion of the stress intensity factors $K_{\mathrm{I}}(s)$, $K_{\mathrm{II}}(s)$ at the tip of the extended crack in powers of s.

We first introduce the expansion of the functional $\mathbb{L}$ defined in the preceding section in powers of s:

$$\mathbb{L}(\varphi, R, C, a, s; \mathscr{T}) \equiv \mathbb{L}^*(\varphi, R, C; \mathscr{T}) + \mathbb{L}^{(1/2)}(\varphi, R, C, a; \mathscr{T})\sqrt{s} + O(s). \tag{19}$$

The fact that the second term of the expansion of $\mathbb{L}$ is proportional to $\sqrt{s}$ instead of s as one would expect at first sight can be justified in the same way as in the case without contact (see Leblond, 1989). The argument a in the function $\mathbb{L}^*$ is discarded from now on. Indeed $\mathbf{K}^*$ has been shown in the preceding section to be a function only of the kink angle φ and the stress intensity factor K_{II} of the initial closed crack, which does depend on the loading $\mathscr{T}$ and the curvature C of that crack (whence the presence of C in the functional $\mathbb{L}^*$) but is obviously independent of the curvature parameter a of the future crack extension.

Expanding eqn (7) in powers of s using eqn (19), one easily sees that the new functional $\mathbb{L}^{(1/2)}$, just like $\mathbb{L}$ and $\mathbb{L}^*$, is positively homogeneous of degree 1 with respect to the loading:

$$\mathbb{L}^{(1/2)}(\varphi, R, C, a; \lambda\mathscr{T}) = \lambda\mathbb{L}^{(1/2)}(\varphi, R, C, a; \mathscr{T})\ (\forall\lambda > 0). \tag{20}$$

Similarly, expanding eqn (8) in powers of s using eqn (19), we get

$$\mathbb{L}^*(\varphi, \lambda R, C/\lambda; \mathscr{T}) + \mathbb{L}^{(1/2)}\left(\varphi, \lambda R, C/\lambda, a/\sqrt{\lambda}; \mathscr{T}\right)\sqrt{\lambda s} + O(s)$$

$$= \sqrt{\lambda}\left[\mathbb{L}^*(\varphi, R, C; \mathscr{T}) + \mathbb{L}^{(1/2)}(\varphi, R, C, a; \mathscr{T})\sqrt{s} + O(s)\right]\ (\forall\lambda > 0);$$

identifying terms of order $\sqrt{s}$ in this equation, one finds that the functional $\mathbb{L}^{(1/2)}$ verifies the following 'positive homogeneity' property with respect to its geometric arguments:

$$\mathbb{L}^{(1/2)}\left(\varphi, \lambda R, C/\lambda, a/\sqrt{\lambda}; \mathscr{T}\right) = \mathbb{L}^{(1/2)}(\varphi, R, C, a; \mathscr{T})\ (\forall\lambda > 0), \tag{21}$$

which differs from its counterparts, eqns (8) and (10), for $\mathbb{L}$ and $\mathbb{L}^*$ by a factor of $\sqrt{\lambda}$.

Inserting eqn (19) into eqn (11), and accounting for eqn (12), one gets

$$\mathbf{K}(s) \equiv \mathbf{K}^* + \mathbf{K}^{(1/2)}\sqrt{s} + O(s) \tag{22}$$

where $\mathbf{K}^*$ is given by eqn (13) and $\mathbf{K}^{(1/2)}$ by the formula

$$\mathbf{K}^{(1/2)} = \mathbb{L}^{(1/2)}\big(\varphi, R, C, a; \mathscr{T}(R)\big). \tag{23}$$

Thus, just like $\mathbf{K}^*$, $\mathbf{K}^{(1/2)}$ depends upon the mechanical fields *only through their values prior to kinking*.

Using eqn (21) with $\lambda = 1/R$, we transform this expression of $\mathbf{K}^{(1/2)}$ into

$$\mathbf{K}^{(1/2)} = \mathbb{L}^{(1/2)}\big(\varphi, 1, RC, \sqrt{R}a; \mathscr{T}(R)\big). \tag{24}$$

In order to now expand this formula, which is valid for all sufficiently small values of R, in powers of this parameter, we first expand the functional $\mathbb{L}^{(1/2)}$ itself (for a fixed, given loading $\mathscr{T}$):

$$\mathbb{L}^{(1/2)}\big(\varphi, 1, RC, \sqrt{R}a; \mathscr{T}\big) = \mathbb{L}^{(1/2)}(\varphi, 1, 0, 0; \mathscr{T}) + \sqrt{R}a\frac{\partial\mathbb{L}^{(1/2)}}{\partial a}(\varphi, 1, 0, 0; \mathscr{T}) + O(R). \tag{25}$$

We also account for the expansion of the loading $\mathscr{T}(R)$, which derives from eqn (3):

$$\mathscr{T}(R) \equiv \frac{K_{\mathrm{II}}}{\sqrt{R}}\big\{\mathbf{f}_{\mathrm{II}}(\theta)\cdot\mathbf{e}_r(\theta)\big\} + T_{\mathrm{I}}\big\{\mathbf{g}_{\mathrm{I}}(\theta)\cdot\mathbf{e}_r(\theta)\big\} + T_{\mathrm{II}}\big\{\mathbf{g}_{\mathrm{II}}(\theta)\cdot\mathbf{e}_r(\theta)\big\} + O\big(\sqrt{R}\big). \tag{26}$$

We now insert eqns (25) and (26) into eqn (24), and expand the result in powers of R using the facts that K_{II} is assumed to be positive and that $\mathbb{L}^{(1/2)}$ is positively homogeneous of degree 1 with respect to the loading $\mathscr{T}$, eqn (20), and therefore its derivatives $\partial\mathbb{L}^{(1/2)}/\partial a$ and $\partial\mathbb{L}^{(1/2)}/\partial\mathscr{T}$ positively homogeneous of degrees 1 and 0 respectively with respect to the same argument. The result reads:

$$\mathbf{K}^{(1/2)} = \frac{K_{\mathrm{II}}}{\sqrt{R}}\mathbb{L}^{(1/2)}\big(\varphi, 1, 0, 0; \{\mathbf{f}_{\mathrm{II}}(\theta)\cdot\mathbf{e}_r(\theta)\}\big) + T_{\mathrm{I}}\frac{\partial\mathbb{L}^{(1/2)}}{\partial\mathscr{T}}\big(\varphi, 1, 0, 0; \{\mathbf{f}_{\mathrm{II}}(\theta)\cdot\mathbf{e}_r(\theta)\}\big)\cdot\{\mathbf{g}_{\mathrm{I}}(\theta)\cdot\mathbf{e}_r(\theta)\}$$

$$+T_{\mathrm{II}}\frac{\partial\mathbb{L}^{(1/2)}}{\partial\mathscr{T}}\big(\varphi, 1, 0, 0; \{\mathbf{f}_{\mathrm{II}}(\theta)\cdot\mathbf{e}_r(\theta)\}\big)\cdot\{\mathbf{g}_{\mathrm{II}}(\theta)\cdot\mathbf{e}_r(\theta)\} + aK_{\mathrm{II}}\frac{\partial\mathbb{L}^{(1/2)}}{\partial a}\big(\varphi, 1, 0, 0; \{\mathbf{f}_{\mathrm{II}}(\theta)\cdot\mathbf{e}_r(\theta)\}\big) + O\big(\sqrt{R}\big)$$

where $(\partial\mathbb{L}^{(1/2)}/\partial\mathscr{T})(\varphi, 1, 0, 0; \mathscr{T})\cdot\mathscr{T}'$ denotes the result of the application of the (linear) operator $(\partial\mathbb{L}^{(1/2)}/\partial\mathscr{T})(\varphi, 1, 0, 0; \mathscr{T})$ on the traction field $\mathscr{T}'$. Now the left-hand side here is independent of R by definition. It follows that the divergent term proportional to $1/\sqrt{R}$ in the right-hand side must necessarily be zero. Taking the limit $R\to 0$ in the above expression, we therefore obtain

$$\mathbf{K}^{(1/2)} = T_{\mathrm{I}}\frac{\partial\mathbb{L}^{(1/2)}}{\partial\mathscr{T}}\big(\varphi, 1, 0, 0; \{\mathbf{f}_{\mathbf{II}}(\theta)\cdot\mathbf{e}_r(\theta)\}\big)\cdot\{\mathbf{g}_{\mathrm{I}}(\theta)\cdot\mathbf{e}_r(\theta)\}$$

$$+T_{\mathrm{II}}\frac{\partial\mathbb{L}^{(1/2)}}{\partial\mathscr{T}}\big(\varphi, 1, 0, 0; \{\mathbf{f}_{\mathrm{II}}(\theta)\cdot\mathbf{e}_r(\theta)\}\big)\cdot\{\mathbf{g}_{\mathrm{II}}(\theta)\cdot\mathbf{e}_r(\theta)\} + aK_{\mathrm{II}}\frac{\partial\mathbb{L}^{(1/2)}}{\partial a}\big(\varphi, 1, 0, 0; \{\mathbf{f}_{\mathrm{II}}(\theta)\cdot\mathbf{e}_r(\theta)\}\big), \tag{27}$$

or, in component form,

$$\begin{cases} K_{\mathrm{I}}^{(1/2)} = \tilde{G}_{\mathrm{I,I}}(\varphi)T_{\mathrm{I}} + \tilde{G}_{\mathrm{I,II}}(\varphi)T_{\mathrm{II}} + a\tilde{H}_{\mathrm{I,II}}(\varphi)K_{\mathrm{II}} \\ K_{\mathrm{II}}^{(1/2)} = \tilde{G}_{\mathrm{II,I}}(\varphi)T_{\mathrm{I}} + \tilde{G}_{\mathrm{II,II}}(\varphi)T_{\mathrm{II}} + a\tilde{H}_{\mathrm{II,II}}(\varphi)K_{\mathrm{II}} \end{cases} \tag{28}$$

where $\tilde{G}_{\mathrm{I,I}}(\varphi)$, $\tilde{G}_{\mathrm{I,II}}(\varphi)$, $\tilde{G}_{\mathrm{II,I}}(\varphi)$, $\tilde{G}_{\mathrm{II,II}}(\varphi)$, $\tilde{H}_{\mathrm{I,II}}(\varphi)$, $\tilde{H}_{\mathrm{II,II}}(\varphi)$ are functions depending only on the kink angle φ. Again, this result is to be compared to that obtained by Leblond (1989) in the absence of contact, namely[3]

$$\begin{cases} K_{\mathrm{I}}^{(1/2)} = G_{\mathrm{I}}(\varphi)T + a\big[H_{\mathrm{I,I}}(\varphi)K_{\mathrm{I}} + H_{\mathrm{I,II}}(\varphi)K_{\mathrm{II}}\big] \\ K_{\mathrm{II}}^{(1/2)} = G_{\mathrm{II}}(\varphi)T + a\big[H_{\mathrm{II,I}}(\varphi)K_{\mathrm{I}} + H_{\mathrm{II,II}}(\varphi)K_{\mathrm{II}}\big] \end{cases} \tag{29}$$

where $T\equiv T_{\mathrm{I}}$ and the $G_p(\varphi)$ and $H_{p,q}(\varphi)$ are analogous functions depending only on φ. Again, all these functions are *universal* in the sense that in both cases (with or without contact initially), they apply to any situation, whatever the geometry and the loading.

5. Practical method for calculating the functions $\tilde{\mathbf{F}}_{p,\mathrm{II}}(\boldsymbol{\varphi})$

Although the above reasonings have evidenced the *existence* of the universal functions $\tilde{F}_{p,\mathrm{II}}(\varphi)$, $\tilde{G}_{p,q}(\varphi)$ and $\tilde{H}_{p,\mathrm{II}}(\varphi)$, they have not provided the *values* of these functions. We shall now see how to calculate these values numerically by the finite element method. The simplest case of the functions $\tilde{F}_{p,\mathrm{II}}(\varphi)$ is envisaged here. The methods for calculating the functions $\tilde{G}_{p,q}(\varphi)$ and $\tilde{H}_{p,\mathrm{II}}(\varphi)$ are basically similar to

[3] The parameter a is noted a^* in the work of Leblond (1989).

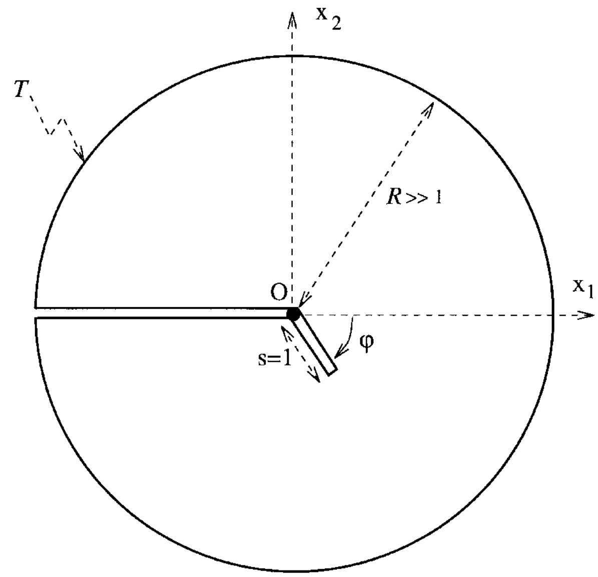

Fig. 5. The geometry used to calculate the various universal functions.

that for calculating the functions $\tilde{F}_{p,\mathrm{II}}(\varphi)$ but involve several additional technical details, and for this reason are relegated to Appendices A and B.

Our starting point consists of eqns (16) and (17), which imply that[4]

$$\tilde{F}_{p,\mathrm{II}}(\varphi) = \mathbb{L}_p^*\big(\varphi, 1, 0; \{\mathbf{f}_{\mathrm{II}}(\theta)\cdot\mathbf{e}_r(\theta)\}\big) \equiv \lim_{s\to 0}\mathbb{L}_p\big(\varphi, 1, 0, 0, s; \{\mathbf{f}_{\mathrm{II}}(\theta)\cdot\mathbf{e}_r(\theta)\}\big)$$

where the definition of $\mathbb{L}^*$ as the limit of $\mathbb{L}$ for $s\to 0$ has been used. We now use the homogeneity property (8) of $\mathbb{L}$ with $\lambda = 1/s$ to transform the preceding equation into

$$\tilde{F}_{p,\mathrm{II}}(\varphi) = \lim_{s\to 0}\sqrt{s}\,\mathbb{L}_p\left(\varphi, \frac{1}{s}, 0, 0, 1; \{\mathbf{f}_{\mathrm{II}}(\theta)\cdot\mathbf{e}_r(\theta)\}\right).$$

We now rewrite s as $1/\mathscr{R}$ and use eqn (7) with $\lambda = \sqrt{1/\mathscr{R}}$ to finally get

$$\tilde{F}_{p,\mathrm{II}}(\varphi) = \lim_{\mathscr{R}\to+\infty}\mathbb{L}_p\left(\varphi, \mathscr{R}, 0, 0, 1; \frac{1}{\sqrt{\mathscr{R}}}\{\mathbf{f}_{\mathrm{II}}(\theta)\cdot\mathbf{e}_r(\theta)\}\right)$$

$$\equiv \lim_{\mathscr{R}\to+\infty}\mathbb{L}_p\left(\varphi, \mathscr{R}, C=0, a=0, s=1; \frac{1}{\sqrt{\mathscr{R}}}\{\mathbf{f}_{\mathrm{II}}(\theta)\cdot\mathbf{e}_r(\theta)\}\right). \quad (30)$$

The physical meaning of this equation is as follows. Let us consider a circular disk of centre O, of radius $\mathscr{R}\to+\infty$ containing a *straight* ($C = 0$) edge crack endowed with a *straight* ($a = 0$) extension of *unit* ($s = 1$) length making an angle φ with the principal branch (Fig. 5). Then $\tilde{F}_{p,\mathrm{II}}(\varphi)$ is the *p-th stress intensity factor at the tip of the extended crack resulting from the application of the fundamental field* $(1/\sqrt{\mathscr{R}})\{\mathbf{f}_{\mathrm{II}}(\theta)\cdot\mathbf{e}_r(\theta)\}$ *on the boundary of that disk*. This provides a practical way to numerically calculate

[4] Here as in Section 4, we discard the argument a in the functional $\mathbb{L}^*$; see eqn (19) for the meaning of its other arguments.

the functions $\tilde{F}_{p,\mathrm{II}}(\varphi)$, using the finite element method (in elasticity with frictionless contact). The requirement $\mathscr{R} \to +\infty$ means in practice that $\mathscr{R}$ must be much larger than the unit length of the crack extension:

$$\mathscr{R} \gg 1. \tag{31}$$

6. Finite element calculation of the functions $\tilde{F}_{p,\mathrm{II}}(\varphi)$, $\tilde{G}_{p,q}(\varphi)$ and $\tilde{H}_{p,\mathrm{II}}(\varphi)$

All finite element computations are performed using the CASTEM code developed by the French *Commissariat à l'Energie Atomique*. They consider the circular geometry described in Section 5 and Appendices A and B (see Fig. 5). The radius of the disk is taken as $\mathscr{R} = 100$, which is much larger than the unit length of the crack extension, as required by condition (31). The mesh is composed of 2826 linear traingles and bilinear quadrangles and comprises 2477 nodes. It is of course refined near the corner point O and even more so near the tip of the crack extension: the average size of the elements is 10 near the external boundary, 0.1 near the corner point and 0.001 near the extended crack tip. The calculations are performed in plane strain. The values of Young's modulus and Poisson's ratio used are $E = 200{,}000$ MPa and $\nu = 0.3$. In fact, it is easy to see that the functions looked for are independent of the value of Young's modulus, but that they do depend upon that of Poisson's ratio, because the contact conditions involve the displacements which depend on that parameter. The stress intensity factors at the tip of the extended crack, which provide the values of the universal functions looked for, as explained in Section 5 and Appendices A and B, are evaluated using the so-called $G-\theta$ method developed by Destuynder et al. (1983), the accuracy of which is well established.

It will be seen in Section 7 that the widely accepted *principle of local symmetry* of Goldstein and Salganik (1974) imposes that the only physically possible value of the kink angle φ is

$$\varphi = \varphi_0 \simeq -77.3^\circ \tag{32}$$

(for $K_{\mathrm{II}} > 0$, as assumed here; for $K_{\mathrm{II}} < 0$, it is trivial by symmetry that φ would take the value $\varphi = -\varphi_0 \simeq +77.3^\circ$). Therefore a single mesh is constructed, for this value of the kink angle.

It is found numerically that in all calculations required to get the values of $\tilde{F}_{p,\mathrm{II}}(\varphi_0)$, $\tilde{G}_{p,\mathrm{I}}(\varphi_0)$ (*not* $\tilde{G}_{p,\mathrm{II}}(\varphi_0)$) and $\tilde{H}_{p,\mathrm{II}}(\varphi_0)$ (see Section 5 and Appendices A and B), no contact can be observed anywhere between the crack lips, neither on the extension nor on the principal branch: the extended crack is entirely open. This implies that these quantities are exactly the same as in ordinary elastic fracture mechanics without contact, that is (see eqns (17) and (18), and (28) and (29)):

$$\tilde{F}_{\mathrm{I,II}}(\varphi_0) \equiv F_{\mathrm{I,II}}(\varphi_0) \simeq 1.23, \quad \tilde{F}_{\mathrm{II,II}}(\varphi_0) \equiv F_{\mathrm{II,II}}(\varphi_0) \simeq 0; \tag{33}$$

$$\tilde{G}_{\mathrm{I,I}}(\varphi_0) \equiv G_{\mathrm{I}}(\varphi_0) \simeq 1.62, \quad \tilde{G}_{\mathrm{II,I}}(\varphi_0) \equiv G_{\mathrm{II}}(\varphi_0) \simeq 0.49; \tag{34}$$

$$\tilde{H}_{\mathrm{I,II}}(\varphi_0) \equiv H_{\mathrm{I,II}}(\varphi_0) \simeq 0.02, \quad \tilde{H}_{\mathrm{II,II}}(\varphi_0) \equiv H_{\mathrm{II,II}}(\varphi_0) \simeq 1.35. \tag{35}$$

The values of the functions without contact here have been taken from the works of Amestoy (1987) and Amestoy and Leblond (1992). The numerical accuracy on these values is of the order of 0.01.

On the other hand, in the calculation required (see Appendix A) to compute the quantities $\tilde{G}_{p,\mathrm{II}}(\varphi_0)$ (which do not exist in ordinary elastic fracture mechanics without contact, see eqn (29)), contact occurs between the crack lips, on the principal branch, away from the corner point O (see Fig. 6), when the

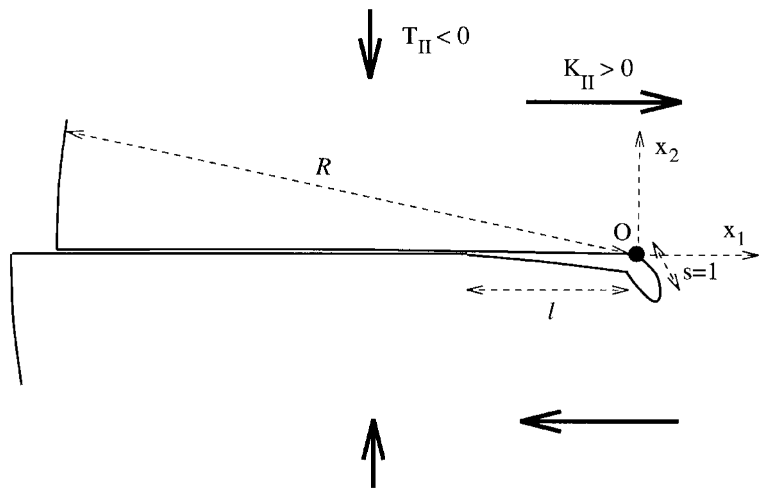

Fig. 6. Schematic deformation of the crack in the calculation required to get the quantities $\tilde{G}_{p,\mathrm{II}}(\varphi_0)$.

loading applied on the boundary of the disk is $\{\mathbf{f}_{\mathrm{II}}(\theta)\cdot\mathbf{e}_r(\theta)\} + T_{\mathrm{II}}\{\mathbf{g}_{\mathrm{II}}(\theta)\cdot\mathbf{e}_r(\theta)\}$ (with $T_{\mathrm{II}} < 0$, $T_{\mathrm{II}} \to 0$, see Appendix A). This is quite normal since far away from the open crack extension, the negative stress $T_{\mathrm{II}} \equiv \sigma_{22}$ (where the indices refer to the frame Ox_1x_2 'adapted' to the crack prior to kinking) tends to close the principal branch of the crack.

This phenomenon puts extra restrictions on the envisageable values of both $\mathscr{R}$ and T_{II}. Indeed, let us denote l the length of that part (the 'separation zone') of the principal branch of the crack where contact does not occur (see Fig. 6). The fact that T_{II} must be 'small' means that its effect must be small in the vicinity of the crack extension. Hence the contact it creates between the lips of the principal branch of the crack must occur far from the crack extension; in other words, the length l of the separation zone on the principal branch of the crack must be large as compared to the unit length of the crack extension:

$$l \gg 1. \tag{36}$$

This condition depends upon the values of both $\mathscr{R}$ and T_{II}. Further, since asymptotically, far away from the point O, the crack lips must be in contact because of the presence of T_{II}, the radius $\mathscr{R}$, which of course cannot be truly infinite, must at least be large enough for the length of the contact zone to be comparable to it; in other words, the length l of the separation zone must be small as compared to it:

$$l \ll \mathscr{R}. \tag{37}$$

This condition also depends on both $\mathscr{R}$ and T_{II}.

On trial and error, it is found that $T_{\mathrm{II}} = -0.065$ is a good value. Indeed it satisfies conditions (A7) of Appendix A. Furthermore it generates a contact zone on the principal branch of the crack of length $\mathscr{R} - l \simeq 89$, that is, a separation zone of length $l \simeq 11$. This value satisfies the two conditions (36), (37) (with $\mathscr{R} = 100$). Using the procedure described in Appendix A, one then finds the following values for the quantities $\tilde{G}_{p,\mathrm{II}}(\varphi_0)$:

$$\tilde{G}_{\mathrm{I,II}}(\varphi_0) \simeq 1.2, \quad \tilde{G}_{\mathrm{II,II}}(\varphi_0) \simeq -2.4. \tag{38}$$

The accuracy here is not known with precision, but is certainly not as good as that for the other quantities which are identical to their counterparts without contact.

7. Conclusions concerning the path followed by the crack

All necessary elements are now gathered to discuss the beginning of the propagation path of the crack. Since it is assumed to be open, at least in the vicinity of its tip, after the initial kink, it is reasonable to adopt, as a propagation criterion for prediction of the crack path, the *principle of local symmetry* of Goldstein and Salganik (1974). This principle is well-known and widely accepted, and convincing theoretical arguments in its favor have been provided by Amestoy (1987). It stipulates that the path followed by the crack is such that the second stress intensity factor $K_{\mathrm{II}}(s)$ at the tip of the extended crack is identically zero all along this path. This means that all terms in the expansion of $K_{\mathrm{II}}(s)$ in powers of the crack extension length s must be zero; in particular, by eqn (22), one must have

$$K^*_{\mathrm{II}} = 0; \quad K^{(1/2)}_{\mathrm{II}} = 0. \tag{39}$$

Combining condition (39_1) with eqn (17_2), with $K_{\mathrm{II}} > 0$ and therefore $\neq 0$, one sees that the kink angle φ must satisfy the equation

$$\tilde{F}_{\mathrm{II,II}}(\varphi) \equiv F_{\mathrm{II,II}}(\varphi) = 0;$$

using the values of the function $F_{\mathrm{II,II}}$ provided by Amestoy (1987) and Amestoy and Leblond (1992), one concludes that the solution of this equation is given by eqn (32). Thus the *kink angle is completely determined independently of the (positive) value of* K_{II}.[5] The difference with respect to the case where the crack is open prior to the kink is that in the latter case, there are *two* initial stress intensity factors K_{I} and K_{II}, so that, by eqn (18_2), eqn (39_1) reads

$$F_{\mathrm{II,I}}(\varphi)K_{\mathrm{I}} + F_{\mathrm{II,II}}(\varphi)K_{\mathrm{II}} = 0 \Longrightarrow \frac{F_{\mathrm{II,I}}(\varphi)}{F_{\mathrm{II,II}}(\varphi)} = -\frac{K_{\mathrm{II}}}{K_{\mathrm{I}}},$$

which determines the kink angle φ as a *continuous function of the ratio* $K_{\mathrm{II}}/K_{\mathrm{I}}$. In other words, if the crack is initially open, there is an extra ‘degree of freedom’ (the ratio $K_{\mathrm{II}}/K_{\mathrm{I}}$) that generates a continuous range of possible kink angles which does not exist if the crack is initially closed.

It should be noted that use of any of the other criteria which have been proposed in the literature, such as the maximum energy release rate criterion, etc., would lead to the same qualitative conclusion, namely that the kink angle is uniquely determined if the crack is initially closed (although the value predicted would be slightly different; for instance it would be $-75.8°$ for the maximum energy release rate criterion, instead of $-77.3°$). Again, this is because these criteria give the value of φ as a function of the ratio $K_{\mathrm{II}}/K_{\mathrm{I}}$, the value of which is fixed to infinity in the presence of initial crack closure (since $K_{\mathrm{I}} \equiv 0$ then).

As mentioned above, what precedes concerns the case where $K_{\mathrm{II}} > 0$. If $K_{\mathrm{II}} < 0$, it is clear by symmetry that the value of the kink angle that will lead to an open crack extension and a zero K^*_{II} will be $-\varphi_0 = +77.3°$. Thus there are in fact two possible values for the kink angle φ: $-77.3°$ if $K_{\mathrm{II}} > 0$ and $+77.3°$ if $K_{\mathrm{II}} < 0$.

Fatigue experiments in mode II for an initially closed crack have been recently performed by Pinna

[5] But in fact the value of K_{II} is not arbitrary, since it must be such that propagation of the crack occurs.

(1997). He observed that after an initial phase where the crack remains straight and closed and propagates in pure mode II, it finally kinks. The propagation path is often quite chaotic at a microscopic scale, but it is rather regular at the macroscopic scale. The (macroscopic) kink angle amounts to about 70°, which is in acceptable agreement with the above prediction.

Let us come back to our theoretical analysis of the crack path, supposing again that $K_{\mathrm{II}} > 0$, and now consider the condition imposed by eqn (39_2). By eqn (28_2), it reads

$$\tilde{G}_{\mathrm{II,I}}(\varphi_0)T_{\mathrm{I}} + \tilde{G}_{\mathrm{II,II}}(\varphi_0)T_{\mathrm{II}} + a\tilde{H}_{\mathrm{II,II}}(\varphi_0)K_{\mathrm{II}} = 0$$

$$\Leftrightarrow G_{\mathrm{II}}(\varphi_0)T_{\mathrm{I}} + \tilde{G}_{\mathrm{II,II}}(\varphi_0)T_{\mathrm{II}} + aH_{\mathrm{II,II}}(\varphi_0)K_{\mathrm{II}} = 0$$

$$\Leftrightarrow a = -\frac{G_{\mathrm{II}}(\varphi_0)T_{\mathrm{I}} + \tilde{G}_{\mathrm{II,II}}(\varphi_0)T_{\mathrm{II}}}{H_{\mathrm{II,II}}(\varphi_0)K_{\mathrm{II}}} \simeq \frac{1}{K_{\mathrm{II}}}(-0.36T_{\mathrm{I}} + 1.8T_{\mathrm{II}}) \tag{40}$$

where eqns (34_2), (35_2), (38_2) have been used. This formula yields the value of the initial curvature parameter a of the crack extension, as a function of the initial stress intensity factor K_{II} and non-singular stresses T_{I} and T_{II}.

It is interesting to see whether or not the signs of the kink angle φ_0 and the curvature parameter a are identical. Indeed, if they are not, it means that after the initial kink, the crack tends to come back to its original direction. On the contrary, if they are, the crack tends to further deviate from its original direction after the kink. Clearly, this behavior depends on the values of the initial non-singular stresses T_{I}, T_{II}.

Let us first investigate the effect of T_{I}, assuming $T_{\mathrm{II}} = 0$. If T_{I} is positive, by eqn (40_2), a is negative, hence its sign is the same as that of φ_0, and therefore the crack tends to further deviate from its initial direction after the initial kink. On the other hand, if T_{I} is negative, the crack tends to come back to its initial direction after the kink. These conclusions concerning the effect of T_{I} are similar to those arrived at by Cotterell and Rice (1980), but in a somewhat different context: here the crack lips are in contact initially and the kink angle is large; in Cotterell and Rice's (1980) study, there was no contact and the kink angle was small, because $|K_{\mathrm{II}}|$ was much smaller than K_{I}.

Let us now examine the effect of $T_{\mathrm{II}} < 0$, assuming $T_{\mathrm{I}} = 0$. Eqn (40_2) implies then $a < 0$. Hence the signs of φ_0 and a are the same, which means that the effect of the second (negative) non-singular stress is to make the crack extension further deviate from its original direction.

These conclusions concerning the effect of the initial non-singular stresses were arrived at by assuming $K_{\mathrm{II}} > 0$. However, it is easy to see by simple symmetry considerations that they still hold for $K_{\mathrm{II}} < 0$. In brief, whatever the sign of K_{II}:

- *For* $T_{\mathrm{II}} = 0$, *the crack tends to further deviate from its initial direction if* $T_{\mathrm{I}} > 0$, *and to come back to it if* $T_{\mathrm{I}} < 0$[6].
- *For* $T_{\mathrm{I}} = 0$ *and* $T_{\mathrm{II}} < 0$ (which is necessary for the crack to be closed initially), *the crack tends to further deviate from its original direction.*

[6] It is recalled that T_{I} represents a simple tension or compression in the direction parallel to the original tangent to the crack at its initial tip O, whereas $T_{\mathrm{II}} < 0$ represents a simple compression in the perpendicular direction.

The latter conclusion is completely specific to the case where contact initially occurs between the crack lips, since there is no such thing as T_{II} in the absence of contact.

Unfortunately, no experiments concerning the influence of T_{II} upon the crack path for an initially closed crack seem available in the literature. Such experiments would not be difficult to perform: it would suffice to load some precracked axisymmetric specimen in combined compression and torsion. Without anticipating too much the results of such experiments, one may reasonably conjecture that they will most probably confirm our theoretical conclusion that the effect of $T_{\mathrm{II}}(<0)$ is to enhance the deviation of the crack from its original direction after the initial kink. Indeed, assuming for instance $K_{\mathrm{II}} > 0$ and looking at Fig. 6, one easily sees that even with a zero $K_{\mathrm{II}}(s)$ just after the kink (as stipulated by the principle of local symmetry), T_{II} will generate a *positive* $K_{\mathrm{II}}(s)$ as s will increase if propagation occurs in a straight manner, which will in fact induce further deviation (curvature) of the crack extension from the initial direction of the crack.

Acknowledgements

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Appendix A. Method for calculating the functions $\tilde{G}_{p,q}(\varphi)$

Our starting point here consists of eqns (27) and (28). If we consider the functions $\tilde{G}_{p,\mathrm{I}}(\varphi)$ for instance, we see that these equations imply that

$$\begin{aligned}\tilde{G}_{p,\mathrm{I}}(\varphi) &= \frac{\partial \mathbb{L}_p^{(1/2)}}{\partial \mathscr{T}}\left(\varphi, 1, 0, 0; \left\{\mathbf{f}_{\mathrm{II}}(\theta)\cdot\mathbf{e}_r(\theta)\right\}\right)\cdot\left\{\mathbf{g}_{\mathrm{I}}(\theta)\cdot\mathbf{e}_r(\theta)\right\} \\ &= \left[\frac{\partial \mathbb{L}_p^{(1/2)}}{\partial T_{\mathrm{I}}}\left(\varphi, 1, 0, 0; \left\{\mathbf{f}_{\mathrm{II}}(\theta)\cdot\mathbf{e}_r(\theta)\right\} + T_{\mathrm{I}}\left\{\mathbf{g}_{\mathrm{I}}(\theta)\cdot\mathbf{e}_r(\theta)\right\}\right)\right]_{T_{\mathrm{I}}=0} \\ &= \lim_{T_{\mathrm{I}}\to 0}\frac{1}{T_{\mathrm{I}}}\left[\mathbb{L}_p^{(1/2)}\left(\varphi, 1, 0, 0; \left\{\mathbf{f}_{\mathrm{II}}(\theta)\cdot\mathbf{e}_r(\theta)\right\} + T_{\mathrm{I}}\left\{\mathbf{g}_{\mathrm{I}}(\theta)\cdot\mathbf{e}_r(\theta)\right\}\right) - \mathbb{L}_p^{(1/2)}\left(\varphi, 1, 0, 0; \right.\right. \\ &\qquad \left.\left.\left\{\mathbf{f}_{\mathrm{II}}(\theta)\cdot\mathbf{e}_r(\theta)\right\}\right)\right].\end{aligned} \tag{A1}$$

Now consider the quantity $\mathbb{L}_p^{(1/2)}(\varphi, 1, 0, 0; \mathscr{T})$. By the very definition (19) of $\mathbb{L}^{(1/2)}$, one has, for any loading $\mathscr{T}$:

$$\begin{aligned}\mathbb{L}_p^{(1/2)}(\varphi, 1, 0, 0; \mathscr{T}) &= \lim_{s\to 0}\frac{1}{\sqrt{s}}\left[\mathbb{L}_p(\varphi, 1, 0, 0, s; \mathscr{T}) - \mathbb{L}_p^*(\varphi, 1, 0; \mathscr{T})\right] \\ &= \lim_{s\to 0}\left[\mathbb{L}_p\left(\varphi, \frac{1}{s}, 0, 0, 1; \mathscr{T}\right) - \mathbb{L}_p^*\left(\varphi, \frac{1}{s}, 0; \mathscr{T}\right)\right]\end{aligned}$$

(by eqns (8) and (10) with $\lambda = 1/s$)

$$= \lim_{\mathscr{R} \to +\infty} \left[\mathbb{L}_p(\varphi, \mathscr{R}, 0, 0, 1; \mathscr{T}) - \mathbb{L}_p^*(\varphi, \mathscr{R}, 0; \mathscr{T}) \right]$$

(where $\mathscr{R} \equiv 1/s$)

$$\equiv \lim_{\mathscr{R} \to +\infty} \left[\mathbb{L}_p(\varphi, \mathscr{R}, C = 0, a = 0, s = 1; \mathscr{T}) - \mathbb{L}_p^*(\varphi, \mathscr{R}, C = 0; \mathscr{T}) \right]. \tag{A2}$$

Inserting eqn (A2) into eqn (A1), one gets

$$\begin{aligned} \tilde{G}_{p,\mathrm{I}}(\varphi) = \lim_{\mathscr{R} \to +\infty} \lim_{T_\mathrm{I} \to 0} \frac{1}{T_\mathrm{I}} \Big[& \mathbb{L}_p\Big(\varphi, \mathscr{R}, C = 0, a = 0, s = 1; \big\{\mathbf{f}_{\mathrm{II}}(\theta) \cdot \mathbf{e}_r(\theta)\big\} + T_\mathrm{I}\big\{\mathbf{g}_\mathrm{I}(\theta) \cdot \mathbf{e}_r(\theta)\big\}\Big) \\ & - \mathbb{L}_p\big(\varphi, \mathscr{R}, C = 0, a = 0, s = 1; \big\{\mathbf{f}_{\mathrm{II}}(\theta) \cdot \mathbf{e}_r(\theta)\big\}\big) \\ & - \mathbb{L}_p^*\Big(\varphi, \mathscr{R}, C = 0; \big\{\mathbf{f}_{\mathrm{II}}(\theta) \cdot \mathbf{e}_r(\theta)\big\} + T_\mathrm{I}\big\{\mathbf{g}_\mathrm{I}(\theta) \cdot \mathbf{e}_r(\theta)\big\}\Big) \\ & + \mathbb{L}_p^*\big(\varphi, \mathscr{R}, C = 0; \big\{\mathbf{f}_{\mathrm{II}}(\theta) \cdot \mathbf{e}_r(\theta)\big\}\big) \Big]. \end{aligned} \tag{A3}$$

We shall now show that the last two terms of this expression cancel out. The physical interpretation of the expression $\mathbb{L}_p^*(\varphi, \mathscr{R}, C = 0; \mathscr{T})$ is recalled to be as follows. Consider a circular disk of centre O, of radius $\mathscr{R}$, containing a *straight* ($C = 0$) edge crack with an *infinitesimal* extension making an angle φ with the principal branch. Then $\mathbb{L}_p^*(\varphi, \mathscr{R}, C = 0; \mathscr{T})$ represents the p-th stress intensity factor at the tip of the extended crack generated by the loading $\mathscr{T}$ applied on the boundary of the disk. Now adding the traction field $T_\mathrm{I}\{\mathbf{g}_\mathrm{I}(\theta) \cdot \mathbf{e}_r(\theta)\}$ to the traction field $\{\mathbf{f}_{II}(\theta) \cdot \mathbf{e}_r(\theta)\}$ on this boundary does not change the stress intensity factor K_{II} *before* the kink, because it only results then in adding an extra *uniform* stress field $\sigma_{11} = T_\mathrm{I}$ (where the indices refer to the frame Ox_1x_2 'adapted' to the crack prior to kinking, see Fig. 1). Therefore, by eqn (17), it does not change the stress intensity factors just after the kink either:

$$\mathbb{L}_p^*\Big(\varphi, \mathscr{R}, C = 0; \big\{\mathbf{f}_{\mathrm{II}}(\theta) \cdot \mathbf{e}_r(\theta)\big\} + T_\mathrm{I}\big\{\mathbf{g}_\mathrm{I}(\theta) \cdot \mathbf{e}_r(\theta)\big\}\Big) = \mathbb{L}_p^*\big(\varphi, \mathscr{R}, C = 0; \big\{\mathbf{f}_{\mathrm{II}}(\theta) \cdot \mathbf{e}_r(\theta)\big\}\big),$$

which is the announced result.

It follows from there and eqn (A3) that

$$\begin{aligned} \tilde{G}_{p,\mathrm{I}}(\varphi) = \lim_{\mathscr{R} \to +\infty} \lim_{T_\mathrm{I} \to 0} \frac{1}{T_\mathrm{I}} \Big[& \mathbb{L}_p\Big(\varphi, \mathscr{R}, C = 0, a = 0, s = 1; \big\{\mathbf{f}_{\mathrm{II}}(\theta) \cdot \mathbf{e}_r(\theta)\big\} + T_\mathrm{I}\big\{\mathbf{g}_\mathrm{I}(\theta) \cdot \mathbf{e}_r(\theta)\big\}\Big) \\ & - \mathbb{L}_p\big(\varphi, \mathscr{R}, C = 0, a = 0, s = 1; \big\{\mathbf{f}_{\mathrm{II}}(\theta) \cdot \mathbf{e}_r(\theta)\big\}\big) \Big]. \end{aligned} \tag{A4}$$

What this equation says is the following. Let us consider, like in Section 5 above, a circular disk of centre O, of radius $\mathscr{R} \to +\infty$ (in practice verifying condition (31)) containing a straight ($C = 0$) edge crack with a straight ($a = 0$) extension of unit ($s = 1$) length making an angle φ with the principal branch (Fig. 5). Then, to get $\tilde{G}_{p,\mathrm{I}}(\varphi)$, one must compare, using the finite element method, the p-th stress intensity factors at the tip of the extended crack generated by the application of the traction fields $\{\mathbf{f}_{\mathrm{II}}(\theta) \cdot \mathbf{e}_r(\theta)\}$ and $\{\mathbf{f}_{\mathrm{II}}(\theta) \cdot \mathbf{e}_r(\theta)\} + T_\mathrm{I}\{\mathbf{g}_\mathrm{I}(\theta) \cdot \mathbf{e}_r(\theta)\}$, where T_I is 'small', on the boundary of that disk. Since the traction fields $\{\mathbf{f}_{\mathrm{II}}(\theta) \cdot \mathbf{e}_r(\theta)\}$ and $\{\mathbf{g}_\mathrm{I}(\theta) \cdot \mathbf{e}_r(\theta)\}$ are of order unity, the condition to be fulfilled by T_I is

$$|T_\mathrm{I}| \ll 1. \tag{A5}$$

The same reasoning can be made for the calculation of the functions $\tilde{G}_{p,\mathrm{II}}(\varphi)$, with the only difference that T_{II} is subject to the restrictive condition $T_{\mathrm{II}} < 0$, which arises from the hypothesis that the crack is initially closed. The final formula for $\tilde{G}_{p,\mathrm{II}}(\varphi)$ is analogous to eqn (A4) for $\tilde{G}_{p,\mathrm{I}}(\varphi)$ except for this restriction:

$$\tilde{G}_{p,\mathrm{II}}(\varphi) = \lim_{\mathscr{R} \to +\infty} \lim_{T_{\mathrm{II}} \to 0^-} \frac{1}{T_{\mathrm{II}}} \Big[\mathbb{L}_p\Big(\varphi, \mathscr{R}, C = 0, a = 0, s = 1; \big\{\mathbf{f}_{\mathrm{II}}(\theta) \cdot \mathbf{e}_r(\theta)\big\} + T_{\mathrm{II}}\big\{\mathbf{g}_{\mathrm{II}}(\theta) \cdot \mathbf{e}_r(\theta)\big\}\Big)$$

$$-\mathbb{L}_p\big(\varphi, \mathscr{R}, C = 0, a = 0, s = 1; \big\{\mathbf{f}_{\mathrm{II}}(\theta) \cdot \mathbf{e}_r(\theta)\big\}\big)\Big]. \tag{A6}$$

The practical method for computation of the functions $\tilde{G}_{p,\mathrm{II}}(\varphi)$ is the same as that for the functions $\tilde{G}_{p,\mathrm{I}}(\varphi)$, with the sole difference that the traction field $\{\mathbf{f}_{\mathrm{II}}(\theta) \cdot \mathbf{e}_r(\theta)\} + T_{\mathrm{I}}\{\mathbf{g}_{\mathrm{I}}(\theta) \cdot \mathbf{e}_r(\theta)\}$ must be replaced by $\{\mathbf{f}_{\mathrm{II}}(\theta) \cdot \mathbf{e}_r(\theta)\} + T_{\mathrm{II}}\{\mathbf{g}_{\mathrm{II}}(\theta) \cdot \mathbf{e}_r(\theta)\}$. The conditions to be fulfilled by T_{II} are

$$T_{\mathrm{II}} < 0, \quad |T_{\mathrm{II}}| \ll 1. \tag{A7}$$

Appendix B. Method for calculating the functions $H_{p,\mathrm{II}}(\varphi)$

Eqns (27) and (28) imply that

$$\tilde{H}_{p,\mathrm{II}}(\varphi) = \frac{\partial \mathbb{L}_p^{(1/2)}}{\partial a}\big(\varphi, 1, 0, 0; \big\{\mathbf{f}_{\mathrm{II}}(\theta) \cdot \mathbf{e}_r(\theta)\big\}\big)$$

$$= \lim_{a \to 0} \frac{1}{a}\Big[\mathbb{L}_p^{(1/2)}\big(\varphi, 1, 0, a; \big\{\mathbf{f}_{\mathrm{II}}(\theta) \cdot \mathbf{e}_r(\theta)\big\}\big) - \mathbb{L}_p^{(1/2)}\big(\varphi, 1, 0, 0; \big\{\mathbf{f}_{\mathrm{II}}(\theta) \cdot \mathbf{e}_r(\theta)\big\}\big)\Big]. \tag{B1}$$

Now consider the quantity $\mathbb{L}_p^{(1/2)}(\varphi, 1, 0, a; \mathscr{T})$. By eqn (19), for any loading $\mathscr{T}$:

$$\mathbb{L}_p^{(1/2)}(\varphi, 1, 0, a; \mathscr{T}) = \lim_{s \to 0} \frac{1}{\sqrt{s}}\Big[\mathbb{L}_p(\varphi, 1, 0, a, s; \mathscr{T}) - \mathbb{L}_p^*(\varphi, 1, 0; \mathscr{T})\Big]$$

$$= \lim_{s \to 0}\Big[\mathbb{L}_p\Big(\varphi, \frac{1}{s}, 0, a\sqrt{s}, 1; \mathscr{T}\Big) - \mathbb{L}_p^*\Big(\varphi, \frac{1}{s}, 0; \mathscr{T}\Big)\Big]$$

(by eqns (8) and (10) with $\lambda = 1/s$)

$$= \lim_{\mathscr{R} \to +\infty}\Big[\mathbb{L}_p\big(\varphi, \mathscr{R}, 0, a/\sqrt{\mathscr{R}}, 1; \mathscr{T}\big) - \mathbb{L}_p^*(\varphi, \mathscr{R}, 0; \mathscr{T})\Big] \tag{B2}$$

(where $\mathscr{R} \equiv 1/s$).

Inserting eqn (B2) into eqn (B1), we obtain

$$\tilde{H}_{p,\mathrm{II}}(\varphi) = \lim_{\mathscr{R} \to +\infty} \lim_{a \to 0} \frac{1}{a}\Big[\mathbb{L}_p\Big(\varphi, \mathscr{R}, 0, a/\sqrt{\mathscr{R}}, 1; \big\{\mathbf{f}_{\mathrm{II}}(\theta) \cdot \mathbf{e}_r(\theta)\big\}\Big) - \mathbb{L}_p\big(\varphi, \mathscr{R}, 0, 0, 1; \big\{\mathbf{f}_{\mathrm{II}}(\theta) \cdot \mathbf{e}_r(\theta)\big\}\big)\Big]$$

$$= \lim_{\mathscr{R} \to +\infty} \lim_{a' \to 0} \frac{1}{\sqrt{\mathscr{R}}a'}\Big[\mathbb{L}_p\big(\varphi, \mathscr{R}, 0, a', 1; \big\{\mathbf{f}_{\mathrm{II}}(\theta) \cdot \mathbf{e}_r(\theta)\big\}\big) - \mathbb{L}_p\big(\varphi, \mathscr{R}, 0, 0, 1; \big\{\mathbf{f}_{\mathrm{II}}(\theta) \cdot \mathbf{e}_r(\theta)\big\}\big)\Big]$$

(where $a' \equiv a/\sqrt{\mathscr{R}}$)

$$= \lim_{\mathscr{R}\to+\infty} \lim_{a'\to 0} \frac{1}{a'}\left[\mathbb{L}_p\left(\varphi, \mathscr{R}, 0, a', 1; \frac{1}{\sqrt{\mathscr{R}}}\left\{\mathbf{f}_{\mathrm{II}}(\theta)\cdot\mathbf{e}_r(\theta)\right\}\right) - \mathbb{L}_p\left(\varphi, \mathscr{R}, 0, 0, 1; \frac{1}{\sqrt{\mathscr{R}}}\left\{\mathbf{f}_{\mathrm{II}}(\theta)\cdot\mathbf{e}_r(\theta)\right\}\right)\right]$$

(by eqn (7) with $\lambda = 1/\sqrt{\mathscr{R}}$). It follows that if we re-note a' as a,

$$\tilde{H}_{p,\mathrm{II}}(\varphi) \equiv \lim_{\mathscr{R}\to+\infty} \lim_{a\to 0} \frac{1}{a}\left[\mathbb{L}_p\left(\varphi, \mathscr{R}, C=0, a, s=1; \frac{1}{\sqrt{\mathscr{R}}}\left\{\mathbf{f}_{\mathrm{II}}(\theta)\cdot\mathbf{e}_r(\theta)\right\}\right)\right.$$
$$\left. -\mathbb{L}_p\left(\varphi, \mathscr{R}, C=0, a=0, s=1; \frac{1}{\sqrt{\mathscr{R}}}\left\{\mathbf{f}_{\mathrm{II}}(\theta)\cdot\mathbf{e}_r(\theta)\right\}\right)\right]. \tag{B3}$$

Eqn (B3) says that the functions $\tilde{H}_{p,\mathrm{II}}(\varphi)$ can be calculated (by the finite element method) in the following way. Consider, like in Section 5 and Appendix A, a circular disk of centre O, radius $\mathscr{R}\to+\infty$ (in practice satisfying condition (31)) containing a straight ($C = 0$) edge crack endowed with an extension of unit ($s = 1$) length making an initial angle φ with the principal branch. Enforce the loading $(1/\sqrt{\mathscr{R}})\{\mathbf{f}_{\mathrm{II}}(\theta)\cdot\mathbf{e}_r(\theta)\}$ on the boundary of this disk. Then $\tilde{H}_{p,\mathrm{II}}(\varphi)$ can be obtained by comparing the p-th stress intensity factors at the tip of the extended crack in the cases where the crack extension is curved ($a \neq 0$) and where it is not ($a = 0$). The curvature parameter a must be 'small' for the comparison; in practical terms, this means that the distance between the tips of the two crack extensions (curved and straight) must be small as compared to the unit extension length, or in other words (see eqn (1)) that a must satisfy the condition

$$|a| \ll 1. \tag{B4}$$

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